

VLASOV MODEL FOR THE IMPEDANCE OF A ROD-PINCH DIODE*

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Abstract

The rod-pinch diode[1,2] is a cylindrical, pinched-beam diode being developed as a radiography source[3]. The diode consists of a small radius anode rod extending through the hole of an annular cathode. The diode has been operated at 1 to 5 MV with an impedance of 20 to 50 Ω , a FWHM pulse width of 20 to 50 ns, and an anode radius as small as 0.25 cm[1-3]. The diode is designed to run at critical current so that electrons emitted from the cathode flow axially along the anode rod and pinch radially onto the rod tip. Typically, ion emission from the anode is required for propagation of the pinch along the rod. Without ions, the pinch would occur on the anode rod just downstream of the cathode disk. In order to assure that a given diode will be properly designed to run at critical current requires a detailed knowledge of the diode impedance characteristics. Initially, a laminar flow model[4] was developed to describe the rod-pinch diode. Although this model provides considerable insight into diode operation, PIC simulations show that the electron flow is not laminar[2]. The model of [4] was extended to include transverse electron pressure in order to consider the effects of nonlaminar flow[5]. However, a form for the transverse pressure tensor is required to close the equation set in this model and only special forms of the pressure tensor are analytically tractable. Here, a Vlasov model for the diode electron flow is developed using an electron distribution function with properties that are well characterized and directly related to a rod-pinch diode. In this model, the pressure tensor is self-consistently derived.

I. VLASOV MODEL FOR THE ELECTRON FLOW

The geometry used in the rod-pinch diode model is illustrated in Fig. 1. Ions are emitted along the entire length ℓ of the anode rod and flow radially outward while electrons are only emitted from the bottom of the cathode at radius r_c and flow axially in a sheath above the anode. The radius of the anode rod is r_a and the inner edge of the anode sheath is at r_s . There are two distinct regions; the

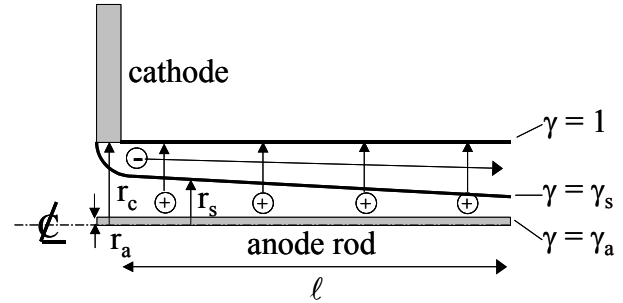


Figure 1. Schematic of rod-pinch diode.

sheath ($r_c \geq r \geq r_s$) contains ion and electrons, while the region between the sheath and the anode ($r_s \geq r \geq r_a$) contains only ions. At $z = \ell$, the electrons pinch onto the end of the rod (not shown). Because $\ell \gg r_c$, $d/dr \gg d/dz$ except at the end of the rod where the pinch occurs. Thus, the model assumes that $d/dz = 0$ and canonical axial momentum, $P_z = p_z - eA_z/c$, is conserved, where p_z is the axial momentum, $A_z(r)$ is the vector potential with $A_z(r_c) = 0$, e is the charge on an electron and c is the speed of light. Energy, $H = [m^2c^4 + (p_r^2 + p_\theta^2 + p_z^2)c^2]^{1/2} - e\phi$, and canonical angular momentum, $P_\theta = rp_\theta$, are also conserved, where p_r and p_θ are the radial and azimuthal momenta, $\phi(r)$ is the electric potential with $\phi(r_c) = 0$, and m is the mass of the electron. For electrons born on the cathode with $P_\theta = 0$, $P_z = 0$, and $H = mc^2$, the electron distribution function is written as

$$f(\vec{p}) = N\delta(P_\theta)\delta(P_z)\delta(H - mc^2) \quad , \quad (1)$$

where N is a constant. Moments of the distribution function can be taken to derive expressions for the electron fluid density $n_e(r)$, axial velocity $V_{ez}(r)$, and pressure $P_{\pi}(r)$ in terms of $A_z(r)$ and $\phi(r)$. In evaluating these integrals, use is made of the identity $\delta(g(x)) = \sum[\delta(x - x_i)/|dg(x)/dx|]$ where the sum is over the zeroes of the function, $g(x_i) = 0$. These expressions are

$$n_e = \frac{N}{cr} \frac{(e\phi + mc^2)}{[(e\phi + mc^2)^2 - e^2 A_z^2 - m^2 c^4]^{1/2}} \quad , \quad (2)$$

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14. ABSTRACT The rod-pinch diode[1,2] is a cylindrical, pinched-beam diode being developed as a radiography source[3]. The diode consists of a small radius anode rod extending through the hole of an annular cathode. The diode has been operated at 1 to 5 MV with an impedance of 20 to 50 Ω, a FWHM pulse width of 20 to 50 ns, and an anode radius as small as 0.25 cm[1-3]. The diode is designed to run at critical current so that electrons emitted from the cathode flow axially along the anode rod and pinch radially onto the rod tip. Typically, ion emission from the anode is required for propagation of the pinch along the rod. Without ions, the pinch would occur on the anode rod just downstream of the cathode disk. In order to assure that a given diode will be properly designed to run at critical current requires a detailed knowledge of the diode impedance characteristics. Initially, a laminar flow model[4] was developed to describe the rod-pinch diode. Although this model provides considerable insight into diode operation, PIC simulations show that the electron flow is not laminar[2]. The model of [4] was extended to include transverse electron pressure in order to consider the effects of nonlaminar flow[5]. However, a form for the transverse pressure tensor is required to close the equation set in this model and only special forms of the pressure tensor are analytically tractable. Here, a Vlasov model for the diode electron flow is developed using an electron distribution function with properties that are well characterized and directly related to a rod-pinch diode. In this model, the pressure tensor is self-consistently derived.		
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II. EQUATIONS FOR ϕ AND A_z

Substituting these results into Poisson's equation and Ampere's law yields

$$V_{ez} = \frac{eA_z c}{(e\phi + mc^2)}, \text{ and} \quad (3)$$

$$P_{rr} = \frac{N}{cr} [(e\phi + mc^2)^2 - e^2 A_z^2 - m^2 c^4]^{1/2}. \quad (4)$$

As a self-consistency check, these results can be shown to satisfy the momentum transfer equation. All other components of the fluid velocity and pressure tensor are identically zero. Note that the pressure P_{rr} given in Eq. (4) vanishes both at the cathode where the electron have zero kinetic energy (and $\phi = A_z = 0$) and at the sheath edge where electrons move only axially. This sheath condition, $P_{rr}(r_s) = 0$, will be needed to obtain a solution. As expected, $P_{rr}(r)$ is found to have a maximum between the cathode and the sheath edge. Also note that it follows from the properties of P_{rr} that n_e is singular at both the cathode and sheath edge and has a minimum between the cathode and sheath edge. Expansion techniques to obtain approximate analytic solutions at these singularities are used to aid in obtaining the numerical solution. Finally, V_{ez} vanishes at the cathode where $A_z = 0$ and, as derived from the sheath condition, increases to $(\gamma_s^2 - 1)^{1/2} c / \gamma_s$ at the sheath edge where $\gamma_s = 1 + e\phi(r_s)/mc^2$ is the electron relativistic factor at the sheath edge.

$$\frac{I}{R} \frac{\partial}{\partial R} \left(R \frac{\partial \gamma}{\partial R} \right) = \frac{J_{ec} \gamma}{R(\gamma^2 - S^2 - 1)^{1/2}} - \frac{J_{ic}}{R(\gamma_a - \gamma)^{1/2}}, \text{ and} \quad (5)$$

$$\frac{I}{R} \frac{\partial}{\partial R} \left(R \frac{\partial S}{\partial R} \right) = \frac{J_{ec} S}{R(\gamma^2 - S^2 - 1)^{1/2}}, \quad (6)$$

where $R = r/r_c$, $\gamma(R) = 1 + e\phi(R)/mc^2$, $S(R) = eA_z(R)/mc^2$, $J_{ec} = 4\pi e^2 N r_c / mc^3$ is the normalized axial electron current density at r_c , $J_{ic} = (4\pi e j_{ic} r_c^2 / mc^3)(M/2mZ)^{1/2}$ is the normalized radial ion current density at r_c , j_{ic} is the radial ion current density at r_c , and M and Z_i are the ion mass and charge state. The ion contribution on the right hand side of Eq. (5) is the same as derived previously[4,5]. In the region below the sheath (i.e., $r_s \geq r \geq r_a$), the same equations apply but with $J_{ec} = 0$. Since $d/dz = 0$ was assumed, the local value of the ion current (through the boundary conditions) determines the axial location z where the solution applies. For example, the full ion

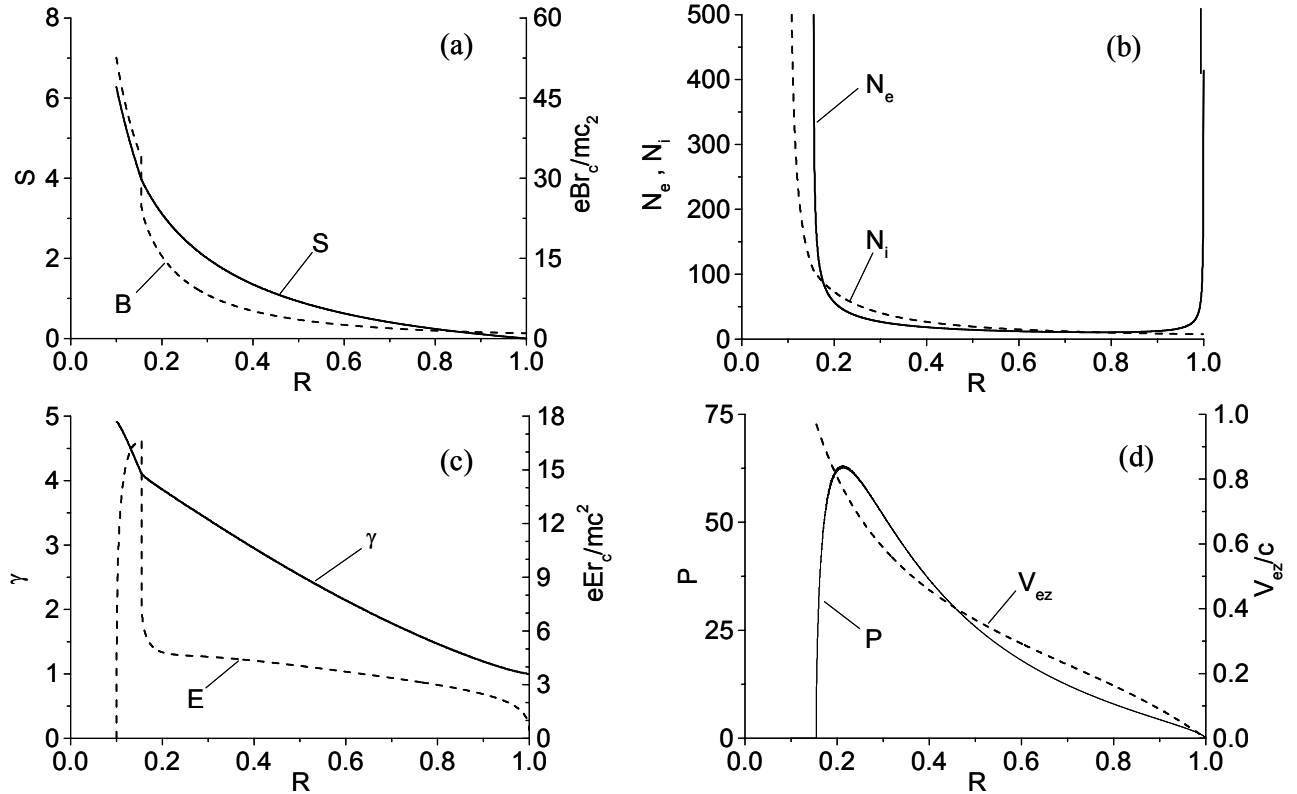


Figure 2. Solution for case with $\gamma_a = 4.9139$ (i.e., $V = 2$ MV), $R_a = r_a/r_c = 0.1$, and $\ell/r_c = 2$ where $R_s = r_s/r_c$ is found to be 0.155. Here, $N_e = 4\pi r_c^2 e c n_e / I_A$, $N_i = 4\pi r_c^2 e Z c n_i / I_A$, and $P = 4\pi e r_c P_{rr} / m c I_A$.

current flows in the rod at the cathode location ($z = 0$) and no ion current flows in the rod at the end of the rod ($z = \ell$). A series of such local solutions provides a global solution. Here, only solutions at the cathode location are considered, so that $z = 0$ and $I_i = 2\pi r_c \ell j_{ic}$, where it is assumed for simplicity that j_{ic} does not vary with z . Solutions are obtained by integrating Eqs. (5) and (6) from the outer boundary in each region (i.e., from r_c and r_a) to the sheath where $R = r_s/r_c = R_s$. Three boundary conditions are known at the cathode where $R = 1$ and two are known at the anode where $R = r_a/r_c = R_a$. These are $\gamma(1) = 1$, $\gamma'(1) = 0$, $S(1) = 0$, $\gamma(R_a) = 1 + eV/mc^2$, and

$\gamma'(R_a) = 0$ where $V = \phi(R_a)$. There are five unknowns J_{ec} , R_s , $S'(1)$, $S(R_a)$, and $S'(R_a)$ which must be guessed to carry out the integrations. The three unknown boundary conditions can be expressed as $-S'(1) = eB(r_c)r_c/mc^2 = 2I_i/I_A$, $S(R_a) = eA_z(r_a)/mc^2$, and $-S'(R_a) = eB(r_a)r_c/mc^2 = 2(I_i + I_e)/I_A R_a$. Here, I_i and I_e are the ion and electron currents and $I_A = mc^3/e = 17$ kA is related to the Alfvén current. Note that J_{ic} , which is needed to integrate Eq. (5), can be written in terms of $S'(1)$ through $J_{ic} = (2I_i r_c / \ell I_A)(M/2Z_i m)^{1/2}$. Valid values for these five unknowns are determined by five conditions that need to be satisfied. These include matching $\gamma(R_s)$, $\gamma'(R_s)$, $S(R_s)$

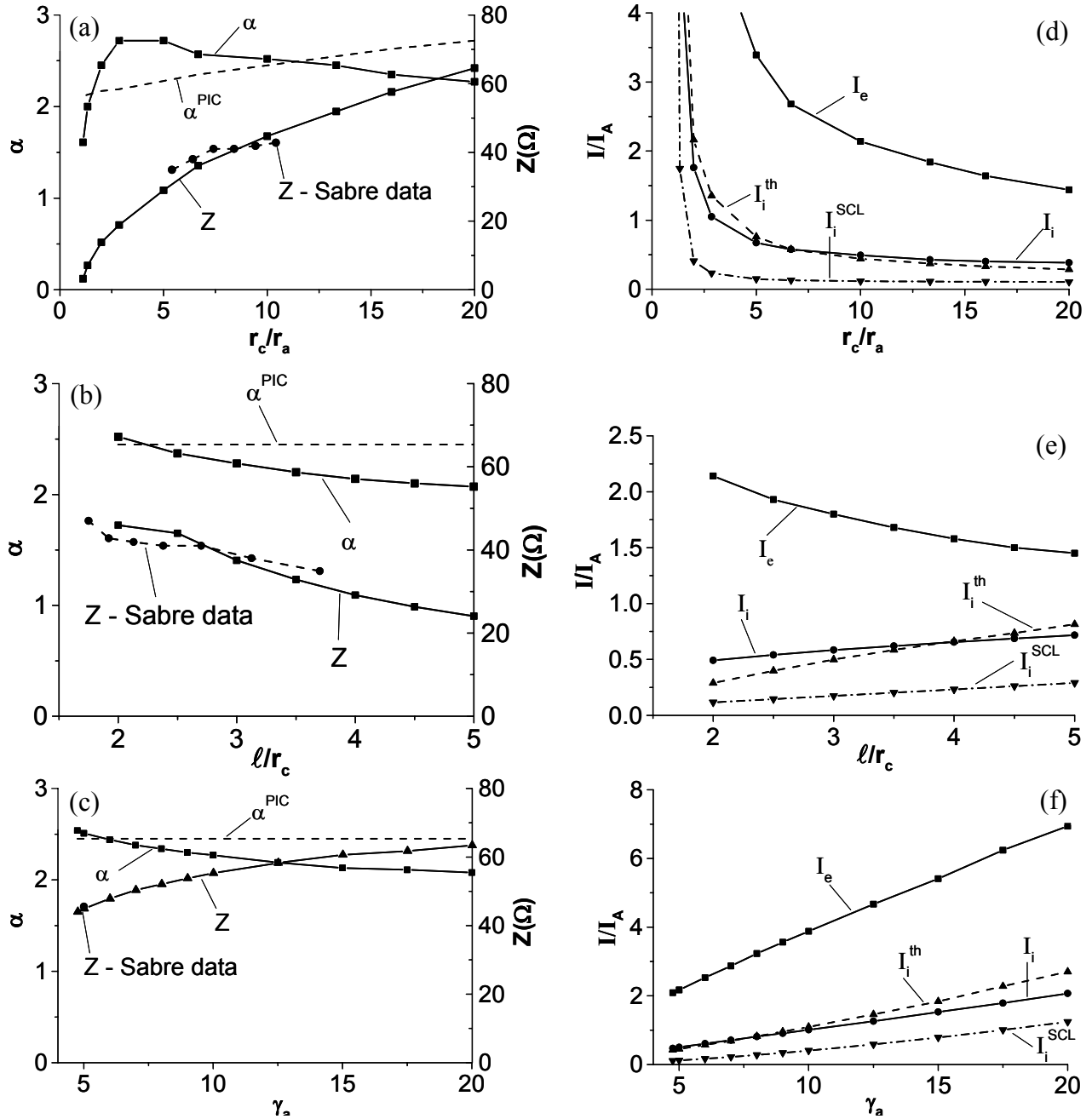


Figure 3. Comparisons of numerical solutions with PIC simulation results, other theoretical predictions, and Sabre data. In (a) and (d), $V = 2$ MV and $\ell/r_c = 2$; in (b) and (e), $V = 2$ MV and $r_c/r_a = 10$; and in (c) and (f), $\ell/r_c = 2$ and $r_c/r_a = 10$.

and $S'(R_s)$ from the two solutions at the sheath and satisfying the sheath condition $[\gamma^2(R_s) - S^2(R_s) - 1] = 0$ (from $P_\pi(R_s) = 0$). Thus, the problem is well posed and the solution provides values for the five unknowns or equivalently for J_{ec} , R_s , $A_z(r_a)$, I_i , and I_e .

III. RESULTS AND DISCUSSION

Results from the numerical solution of Eqs. (5) and (6) are shown in Fig. 2 for the case where $\gamma_a = 4.9139$, $r_c/r_a = 10$, and $\ell/r_c = 2$. Here, the normalized densities are $N_e = 4\pi r_c^2 e n_e / I_A$ and $N_i = 4\pi r_c^2 e Z n_i / I_A$, and the normalized pressure is $P = 4\pi r_c^2 e P_\pi / m c I_A$. Note that, as predicted, the electron density diverges at the cathode and at the edge of the sheath and the electron pressure goes to zero at the cathode and the edge of the sheath. Also, in the region between the anode and the sheath where there are only ions the electric field is highly peaked. Results obtained under various conditions are compared with PIC simulation results[2], other theoretical predictions[2], and data from experiments on the Sabre accelerator at Sandia National Laboratories[3,6] in Fig. 3. Plots of α , Z , I_i , and I_e are shown over a range of values for r_c/r_a , ℓ/r_c , and γ_a . The electrical performance of the diode is characterized by its impedance $Z = V/I$, where $I = I_e + I_i$ is the total diode current. Values of Z obtained from the numerical solutions compare reasonably well with the data obtained on Sabre as shown in Figs. 3a, 3b, and 3c. The diode current has been predicted[2] to vary as $I = \alpha I_{crit} = 8500\alpha(\gamma^2 - 1)^{1/2} / \ln(r_c/r_a)$ where α is a scaling factor. Values of $\alpha = (I_e + I_i)/I_{crit}$ from the numerical solutions are close in value but scale somewhat differently from the factor α^{PIC} found in PIC simulations as illustrated in Figs. 3a, 3b, and 3c. Similarly, the ion current has been theoretically predicted[2] to scale roughly as $I_i^{th} = 1.6 I_e [Z_i m(\gamma + 1)/2M]^{1/2} \ell / (r_c - r_a)$ for $\ell \gg (r_c - r_a)$. Without enhancement from the pinched electron flow, the space-charge-limited ion current $I_i^{SCL} = 7.33 \times 10^{-3} (\gamma_a - 1)^{3/2} \ell / \beta^2 r_c$ would flow. Here, β^2 is given in [7] and the ions are protons. Values of I_e and I_i from the numerical solutions are compared with I_i^{th} (using I_e from the numerical solution in the above equation for I_i^{th}) and I_i^{SCL} in Figs. 3d, 3e, and 3f. As expected, values of I_i found from the numerical solutions are much larger than I_i^{SCL} . The agreement with I_i^{th} is reasonable in absolute value but the scaling is somewhat different.

IV. SUMMARY

A Vlasov model for the diode electron flow in a rod-pinch diode is developed that is derived from a distribution function for the electrons with properties that are well characterized and directly related to the rod-pinch diode. Non-laminar flow is self-consistently included and results in a transverse radial pressure which is defined in

Eq. (4). Results from the model agree reasonable well in magnitude with the data and results from PIC simulations but differences in scaling are observed. In the model, all electrons are born with only their rest energy and zero canonical angular momentum as would be the case in an experiment where all electrons are born at rest on the cathode surface. However, electrons in the model are restricted to have $P_z = 0$, which is only true for electrons born on the bottom of the cathode. Electrons born on the upstream and downstream faces of the cathode would have different P_z . It is also assumed that $d/dz = 0$, so that P_z is conserved. This assumption is also not strictly correct. It is possible that the true 2D nature of the problem including emission from the sides of the cathode (i.e., $P_z \neq 0$ for all electrons) and axial gradients (i.e., $d/dz \neq 0$) in the fields near the cathode and the pinch at the end of the rod can explain the observed differences in scaling. The simulation results displayed in Fig. 3 are from cases where the simulation geometry is set up to model the actual experimental geometry. Simulations will be run in geometries that more closely resemble the case described by the model to study these issues.

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